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BOUNDS FOR PROJECTION CONSTANTS AND 1-SUMMING NORMS

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ABSTRACT. It is shown that projection constants $\lambda(X_n)$ of n-dimensional normed spaces X_n satisfy $\lambda(X_n) \leq \sqrt{n} - c/\sqrt{n}$ where $c > 0$ is a numerical constant. Similarly, the 1-summing norms of (the identity of) X_n can be estimated by $\pi_1(X_n) \geq \sqrt{n} + c/\sqrt{n}$. These estimates are the best possible: for prime n, translation-invariant n-dimensional spaces X_n such that $\lambda(X_n) \geq \sqrt{n} - 2/\sqrt{n}$ and $\pi_1(X_n) \leq \sqrt{n} + 2/\sqrt{n}$ can be constructed. For these spaces Gordon–Lewis constants and distances to Hilbert spaces are large as well: $\mathrm{gl}(X_n) \geq \frac{1}{3}\sqrt{n}$, $d(X_n, l_2^n) = \sqrt{n}$.

1. INTRODUCTION AND RESULTS

The main results of the paper complete the investigation of an upper bound for the projection constants of arbitrary finite-dimensional Banach spaces, which had started with the Kadec–Snobar [KaS] estimate: $\lambda(X_n) \leq \sqrt{n}$, for any n-dimensional space X_n. This estimate was further studied in several papers [KöLL, Kö, KöL, Lew2]. In particular, it has recently been shown by Lewis [Lew2] that $\lambda(X_n) \leq \sqrt{n} - n^{-3/2} 5^{-2(2n+1)}$. Our main theorem gives an essential strengthening, stated in the abstract.

Upper bounds for projection constants are intimately connected to lower bounds for the 1-summing norm of the identity operator. For spaces with enough symmetries these bounds follow from each other by a simple formal argument. In general, it turns out that similar but somewhat simpler calculations than for the projection constants lead to the lower bound for $\pi_1(X_n)$.

Both upper and lower bounds are the best possible, up to terms of higher order. The approach used by König in [Kö], based on some number theoretic results, gives the relevant constructions. The same approach leads to a construction of a finite-dimensional space with large Gordon–Lewis constant. This provides the first deterministic example of such a space.

Our notation essentially follows [TJ]. For completeness, let us recall the main definitions.

By $\mathbf{K}$ we denote the scalar field, either the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$.

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The *relative projection constant* of a (closed) subspace X of a Banach space Y is defined by

$$\lambda(X, Y) := \inf\{\|P\| \mid P: Y \to X \subset Y \text{ is a linear projection onto } X\};$$

the (*absolute*) *projection constant* of a Banach space X is given by

$$\lambda(X) := \sup\{\lambda(X, Y) \mid Y \text{ is a Banach space containing } X \text{ as a subspace}\}.$$

Let $1 \le p \le \infty$ and $u : Y \to Z$ be a (continuous linear) operator between Banach spaces. The *p-factorization norm* of u is defined as

$$\gamma_p(u) := \inf\{\|v\| \, \|w\| \mid \exists \text{ measure space } (\Omega, \mu) \text{ and operators } v : Y \to L_p(\Omega, \mu) \text{ and } w : L_p(\Omega, \mu) \to Z \text{ with } u = wv\},$$

all these expressions being possibly ∞. Any separable Banach space X can be isometrically embedded into l_∞. Thus

$$\lambda(X) = \lambda(X, l_\infty) = \gamma_\infty(\mathrm{Id}_X)$$

where Id_X denotes the identity map on X. For $1 \le p < \infty$ and $u : Y \to Z$ the *p-summing norm* $\pi_p(u)$ is the infimum over all $c > 0$ such that

$$\left(\sum_{i=1}^n \|uy_i\|^p\right)^{1/p} \le c \sup_{\|y^*\|_{Y^*}=1} \left(\sum_{i=1}^n |\langle y^*, y_i\rangle|^p\right)^{1/p}$$

holds for all finite sequences $y_1, \dots, y_n \in Y$. We set $\pi_p(u) = \infty$ if no such c exists. We let $\pi_p(X) = \pi_p(\mathrm{Id}_X)$. It is well known that γ_∞ and π_1 are in trace-duality; i.e., for any finite-dimensional spaces Y and Z and any operator $u : Y \to Z$

$$\gamma_\infty(u) = \sup_{0 \ne v : Z \to Y} \frac{|\mathrm{tr}(vu)|}{\pi_1(v)} \quad \text{and} \quad \pi_1(u) = \sup_{0 \ne v : Z \to Y} \frac{|\mathrm{tr}(vu)|}{\gamma_\infty(v)}.$$

In particular,

$$\lambda(X) = \sup_{0 \ne v : X \to X} \frac{|\mathrm{tr}(v)|}{\pi_1(v)}. \tag{1}$$

This implies that for an n-dimensional space X we get

$$\lambda(X)\pi_1(X) \ge n.$$

A space X is said to have *enough symmetries* if any operator $u : X \to X$ which commutes with all isometries of X is a multiple of the identity map. If X has enough symmetries and $\dim X = n$, then

$$\lambda(X)\pi_1(X) = n; \tag{2}$$

i.e., the sup in (1) is attained by $v = \mathrm{Id}_X$ [GG]. We also need the *Gordon–Lewis constant* of X,

$$\mathrm{gl}(X) := \sup\{\gamma_1(u)/\pi_1(u) \mid 0 \ne u : X \to l_2\},$$

and the *Banach–Mazur distance* from X to Y,

$$d(X, Y) := \inf\{\|u\| \, \|u^{-1}\| \mid u : X \to Y \text{ is an isomorphism}\}.$$

The *unconditional basis constant* $\mathrm{ubc}(x_j)$ of a basis (x_j) of X is the smallest constant $c > 0$ such that for all $(a_j) \subseteq \mathbf{K}$ and $(\varepsilon_j) \subseteq \mathbf{K}$ with $|\varepsilon_j| = 1$,

$$\left\| \sum_j \varepsilon_j a_j x_j \right\| \le c \left\| \sum_j a_j x_j \right\| .$$

One has $\mathrm{gl}(X) \le \mathrm{ubc}(x_j)$ for any basis (x_j) of X and $\mathrm{ubc}(X) := \inf\{\mathrm{ubc}(x_j) \mid (x_j) \text{ a basis of } X\} \le (\dim X)^{1/2}$ (cf., e.g., [TJ, §34]).

If a complex space X is a translation-invariat subspace of the L_∞-space on a compact commutative group and (y_j) is the natural basis of characters in X, then the *Sidon constant* of X, $S(X)$, is the smallest constant $c > 0$ such that for all $(a_j) \subset \mathbf{C}$,

$$\sum_j |a_j| \le c \left\| \sum_j a_j y_j \right\| .$$

It can be shown that $S(X) = \mathrm{ubc}(y_j) \le (\dim X)^{1/2}$ (cf. [Pis], also the proof of Proposition 5).

Our main result for the projection constant states

Theorem 1. *For all $n \in \mathbf{N}$ and any n-dimensional space X_n we have*

$$\lambda(X_n) \le \sqrt{n} - 1/2\sqrt{n} + O(1/n^{3/4}).$$

Remark 1. For real 2-dimensional spaces $X_2^{\mathbf{R}}$ the estimates yield $\lambda(X_2^{\mathbf{R}}) \le 1.378$, while the conjectured extreme value is $\frac{4}{3}$.

We use superscripts $\mathbf{R}$ and $\mathbf{C}$ to distinguish between slightly different estimates in the real and complex cases. A general estimate for the relative projection constant of an n-dimensinal subspace X_n of a N-dimensional space Y_N was given in [KöLL]. It states

$$\begin{aligned} \lambda(X_n, Y_N) \le f(n, N) &:= \sqrt{n} \left(\frac{\sqrt{n} + \sqrt{N-1}\sqrt{N-n}}{N} \right) \\ &\le \sqrt{n} \left(1 - \frac{(\sqrt{n}-1)^2}{2N} \right) . \end{aligned} \tag{3}$$

In the case when X_n has enough symmetries, Theorem 1 can be slightly improved.

Proposition 2. *Let X_n have enough symmetries or* (*at least*) *assume that $\pi_1(X_n)\lambda(X_n) = n$. Then*

$$\lambda(X_n^{\mathbf{R}}) \le f\left(n, \frac{n(n+1)}{2}\right) = \sqrt{n} - \frac{1}{\sqrt{n}} + \frac{2}{n} - O\left(\frac{1}{n^{3/2}}\right),$$

$$\lambda(X_n^{\mathbf{C}}) \le f(n, n^2) = \sqrt{n} - \frac{1}{2\sqrt{n}} + \frac{1}{n} - O\left(\frac{1}{n^{3/2}}\right).$$

Remark 2. For 2- and 3-dimensional real spaces with enough symmetries, $\lambda(X_2^{\mathbf{R}}) \le \frac{4}{3}$ and $\lambda(X_3^{\mathbf{R}}) \le (1+\sqrt{5})/2$. Equality holds for the spaces with the hexagon unit ball and the dodecahedron unit ball, respectively. The π_1-norm is minimal for these spaces.

Conjecture. We conjecture that the estimates of Proposition 2 are true in general (without assuming enough symmetries).

It is well known that $\pi_2(X_n) = \sqrt{n}$ [GG], cf. also [TJ, §9]. This yields an obvious inequality $\sqrt{n} = \pi_2(X_n) \le \pi_1(X_n)$. Techniques of Theorem 1 and Proposition 2 yield the following lower bound for $\pi_1(X_n)$ (without assuming enough symmetries).

Proposition 3. *For all $n \in \mathbf{N}$ and any n-dimensional space X_n we have*

$$\pi_1(X_n^{\mathbf{R}}) \ge n/f\left(n, \frac{n(n+1)}{2}\right) = \sqrt{n} + \frac{1}{\sqrt{n}} - \frac{2}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

$$\pi_1(X_n^{\mathbf{C}}) \ge n/f(n, n^2) = \sqrt{n} + \frac{1}{2\sqrt{n}} - \frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right).$$

Remark 3. The method also works for $\pi_p(X_n)$ if $1 \le p < 2$, yielding $\pi_p(X_n) \ge \sqrt{n} + c_p/\sqrt{n}$. Neglecting terms of higher order $O(1/n)$, c_p may be taken as $2((1/p) - \frac{1}{2})$ in the real case and $(1/p) - \frac{1}{2}$ in the complex case.

The estimates in Theorem 1 and Propositions 2 and 3 are very close to being the best possible, in view of the following constructions.

Proposition 4. (a) *For any prime p and $m \in \mathbf{N}$, set $n := p^m + 1$ and $N := n^2 - n + 1$. One can construct a complex n-dimensional space $X_\infty^n \subset l_\infty^N$ with $\lambda(X_\infty^n)\pi_1(X_\infty^n) = n$ such that*

$$\lambda(X_\infty^n) = f(n, N) = \sqrt{n} - \frac{1}{2\sqrt{n}} + \frac{1}{n} - O\left(\frac{1}{n^{3/2}}\right)$$

and

$$\pi_1(X_\infty^n) = n/f(n, N) = \sqrt{n} + \frac{1}{2\sqrt{n}} - \frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right).$$

(b) *For any $n = 4^m$ and $N := n^2/2$ one can construct a real n-dimensional space $X_\infty^n \subset l_\infty^N$ with*

$$\sqrt{n} - 2/\sqrt{n} \le \lambda(X_\infty^n) \le \sqrt{n} - 1/\sqrt{n}.$$

Thus in (a) the difference between the complex example and the general estimate of Propositions 2 and 3 is only of order

$$f(n, n^2) - f(n, n^2 - n + 1) = O(1/n^{3/2}).$$

The spaces constructed in (a) are translation invariant over finite groups. Another version of this construction leads to spaces with many more "almost extremal" properties. In particular, they would have the gl-constants of maximal order and extremal distance to Hilbert spaces. Moreover, if $(y_j)_{j=1}^n$ is the natural basis of characters in X_∞^n, then $\operatorname{ubc}(y_j) = \sqrt{n}$ or, equivalently, the Sidon constant of X_∞^n is extremal and equal to $\sqrt{n}$.

Proposition 5. *For all odd prime numbers n and $N := (n^2 - 1)/2$ one can construct complex n-dimensional translation-invariant spaces $X_\infty^n \subset l_\infty^N$ with*

(a) $\lambda(X_n) = \sqrt{n} - 2/\sqrt{n} - O(1/n)$,
(b) $\pi_1(X_n) = \sqrt{n} + 2/\sqrt{n} + O(1/n)$,
(c) *the distance to l_2^n is maximal,* $d(X_\infty^n, l_2^n) = \sqrt{n}$,
(d) *the natural basis of characters (y_j) in X_∞^n satisfies* $\mathrm{ubc}(y_j) = \sqrt{n}$,
(e) *the Sidon constant is maximal,* $S(X_\infty^n) = \sqrt{n}$,
(f) *the Gordon–Lewis constant is of maximal order* $\mathrm{gl}(X_\infty^n) \geq \frac{1}{3}\sqrt{n}$.

Remark 4. For any space X_n one has $\mathrm{gl}(X_n) \leq \sup\{\lambda(Y_n) \mid \dim Y_n = n\} \leq \sqrt{n} - c/\sqrt{n}$.

The remaining part of the paper contains mostly the proofs of statements and remarks and is organized as follows. §2 contains proofs of general estimates of Theorem 1 and Propositions 2 and 3. In §3 the constructions of examples are presented and some more related estimates are obtained.

2. Projection constants, summing norms, and ultraspherical polynomials

Theorem 1 and Propositions 2 and 3 are based on a similar argument, which requires several steps presented below. The "proper" proofs of the stated results are provided at the end of the section.

The differences $\pi_1(X_n) - \sqrt{n}$ and $\sqrt{n} - \lambda(X_n)$ will be estimated from below by a suitable double integral I, which is then bounded from below. The basic idea of the first step is due to Lewis [Lew2].

Step 1. Let X be a real or complex n-dimensinal space and define $0 \leq \varepsilon < 1$ by $\pi_1(X) = \sqrt{n}/(1-\varepsilon)$. We want to bound ε from below (like $\varepsilon \geq c/n$) to get a lower estimate for $\pi_1(X)$ and, if $\lambda(X) = n/\pi_1(X)$ as in Proposition 2, an upper bound for $\lambda(X)$. By Pietsch's factorization theorem (cf., e.g., [LT, TJ, §9]), there exist a compact space T and a regular Borel probability measure μ on T, an isometric embedding $i\colon X \to L_\infty(T, \mu)$, and a map $w : L_1(T, \mu) \to L_\infty(T, \mu)$ with $\|w\| = \pi_1(X) = \sqrt{n}/(1-\varepsilon)$ such that the following diagram commutes.

$$\begin{array}{ccccc} X & & \xrightarrow{\quad\quad\mathrm{Id}\quad\quad} & & X \\ i\downarrow & & & & \downarrow i \\ L_\infty(T\mu) & \xrightarrow{I_1} & L_1(T, \mu) & \xrightarrow{w} & L_\infty(T, \mu) \end{array}$$

For $1 \leq p < \infty$, denote by I_p the inclusion $L_\infty(T, \mu) \to L_p(T, \mu)$ and let $X_\infty := i(X)$ and $X_p := I_p i(X)$ be the images of X in $L_\infty(T, \mu)$ and $L_p(T, \mu)$, respectively. Thus for all $g \in X_\infty$,

$$\|g\|_\infty = \|wI_1 g\|_\infty \leq \pi_1(X) \int_T |g(s)|\, d\mu(s). \tag{4}$$

By Lewis [Lew1], the n-dimensional space $X_1 \subset L_1(T, \mu)$ admits a basis $(f_i)_{i=1}^n \subset X$ such that

$$n \int_T f_j(s) \overline{f_k(s)} f(s)^{-1} \, d\mu(s) = \delta_{jk}, \tag{5}$$

where f is the square function $f(s) = (\sum_{j=1}^n |f_j(s)|^2)^{1/2}$. (The result in [Lew1] is stated for real spaces only, but the proof works in the complex case, too.)

Let us make a general remark which will be used several times throughout the argument. If $g_j \in L_\infty(T, \mu)$ for $j = 1, \dots, n$, then, for almost all s,

$$\sum_{j=1}^n |g_j(s)|^2 \leq \left\| \sum_{j=1}^n g_j(s) \overline{g_j} \right\|_\infty . \tag{6}$$

Indeed, this estimate is obvious if the g_j's are continuous. The general case easily follows by approximating a measurable function by a continuous function on sets of measure arbitrarily close to 1.

Let $h_j(s) := \sqrt{n} f_j(s)/f(s)$, $s \in T$, and make the change of density $d\nu(s) = f(s)\, d\mu(s)$. Then

$$\sum_{j=1}^n |h_j(s)|^2 = n \quad \text{and} \quad \int_T h_j(s) \overline{h_k(s)} \, d\nu(s) = \delta_{jk} . \tag{7}$$

Note that ν is a probability measure on T as well, since by (5),

$$\nu(T) = \int_T f(s) \, d\mu(s) = \int_T \left(\sum_{j=1}^n |f_j(s)|^2 / f(s) \right) d\mu(s) = 1 .$$

Let $h(s, t) := \sum_{j=1}^n h_j(s) \overline{h_j(t)}$. Using (5), (6), and (4), we find

$$\begin{aligned} n &= n \left(\int_T f(s) \, d\mu(s) \right)^2 \leq n \int_T |f(s)|^2 \, d\mu(s) \\ &\leq \int_T \left\| \sum_{j=1}^n n f_j(s) \overline{f_j} \right\|_\infty d\mu(s) \\ &\leq \pi_1(X) \int_T \int_T \left| \sum_{j=1}^n n f_j(s) \overline{f_j(t)} \right| d\mu(t) \, d\mu(s) \\ &= \pi_1(X) \int_T \int_T |h(s, t)| \, d\nu(s) \, d\nu(t) . \end{aligned} \tag{8}$$

We choose $\alpha > 0$ close to $\sqrt{n}$ and estimate the mean difference of $(\alpha/n)|h(s, t)|$ and 1 using (7) and (8). We have

$$
\begin{aligned}
I &:= \int_T \int_T \left(\frac{\alpha}{n}|h(s, t)| - 1\right)^2 d\nu(t)\, d\nu(s) \\
&= \int_T \left(\int_T \frac{\alpha^2}{n^2} \left| \sum_{j=1}^n h_j(s)\overline{h_j(t)} \right|^2 d\nu(t) \right) d\nu(s) + 1 \\
&\quad - 2\frac{\alpha}{n} \int_T \int_T |h(s, t)|\, d\nu(t)\, d\nu(s) \\
&\le \frac{\alpha^2}{n} + 1 - 2\frac{\alpha}{\pi_1(X)} = 2\frac{\alpha}{\sqrt{n}}\varepsilon + \left(\frac{\alpha}{\sqrt{n}} - 1\right)^2 .
\end{aligned} \tag{9}
$$

Inequality (9) provides a lower estimate for ε in terms of I. For instance, if $\alpha = \sqrt{n}$, then $\varepsilon \ge \frac{1}{2}I$. However, the final lower bounds for ε, obtained by estimating I from below, will be optimized by taking $\alpha := \sqrt{n+2}$ for real spaces and $\alpha := \sqrt{n+1}$ for complex spaces. This will be of interest for small dimensions n.

Step 2. Before bounding I from below, we give the more elaborate estimate of type (9) in the case of general projection constants, i.e., in the situation of Theorem 1. For this, define $0 \le \varepsilon < 1$ by $\lambda(X) = \sqrt{n}(1 - \varepsilon)$. By (1), there is $v : X \to X$ such that $\lambda(X) = \operatorname{tr}(v)/\pi_1(v)$. We normalize $\pi_1(v) = \sqrt{n}$ so that $\operatorname{tr}(v) = n(1 - \varepsilon)$. Again by Pietsch's factorization theorem, there exist a compact space T and a regular Borel probability measure μ on T, an isometric embedding $i : X \to L_\infty(T, \mu)$, and an "extension" $w : L_1(T, \mu) \to L_\infty(T, \mu)$ of v with $\|w\| = \pi_1(v) = \sqrt{n}$ such that the following diagram commutes.

$$
\begin{array}{ccccc}
X & & \longrightarrow & & X \\
{\scriptstyle i}\big\downarrow & & & & \big\downarrow{\scriptstyle i} \\
L_\infty(T, \mu) & \xrightarrow{I_1} & L_1(T, \mu) & \xrightarrow{w} & L_\infty(T, \mu)
\end{array}
$$

Again, for all $g \in X_\infty := i(X)$ and $\overline{v} := wI_1$, we have

$$
\|\overline{v}(g)\|_\infty \le \pi_1(v) \int_T |g(s)|\, d\mu(s). \tag{4'}
$$

The basis f_j in X_1 satisfying (5) and the functions h_j and h satisfying (7) are constructed as before. Since f_j and $\overline{nf_j/f}$ are μ-biorthogonal, using $(4')$, (5), and (6) we find

$$\begin{aligned}
\operatorname{tr}(v) &= \sum_{j=1}^{n} \langle \overline{v}(f_j), n\overline{f_j}/f\rangle_\mu = \int_T \left(\sum_{j=1}^{n} n\overline{f_j(s)}/f(s)\overline{v}(f_j)(s) \right) d\mu(s) \\
&\le \int_T \left\| \sum_{j=1}^{n} n\overline{f_j(s)}/f(s)\overline{v}(f_j) \right\|_\infty d\mu(s) \\
&\le \pi_1(v) \int_T \int_T \left| \sum_{j=1}^{n} n\overline{f_j(s)}/f(s) f_j(t) \right| d\mu(t)\, d\mu(s) \\
&= \pi_1(v) \int_T \int_T |h(s, t)|\, d\nu(t)\, d\mu(s).
\end{aligned}$$

This and (7) imply, for a fixed $\alpha > 0$ (chosen later or to be close to $\sqrt{n}$),

$$(9') \qquad \begin{aligned}
J &:= \int_T \int_T \left(\frac{\alpha}{n} |h(s, t)| - 1 \right)^2 d\nu(t)\, d\mu(s) \\
&\le 2\frac{\alpha}{\sqrt{n}}\varepsilon + \left(\frac{\alpha}{\sqrt{n}} - 1 \right)^2 =: \delta_1(\varepsilon, n).
\end{aligned}$$

Unfortunately, in contrast to I, the integral J is not symmetric. We symmetrize it using the fact that h is hermitian; i.e., $h(s, t) = \overline{h(t, s)}$. We have

$$\begin{aligned}
\delta_1(\varepsilon, n) \ge J &= \int_T \int_T \left(\frac{\alpha}{n} |h(s, t)| - 1 \right)^2 \frac{f(t) + f(s)}{2}\, d\mu(t)\, d\mu(s) \\
&\ge \int_T \int_T \left(\frac{\alpha}{n} |h(s, t)| - 1 \right)^2 \sqrt{f(t)}\, d\mu(t) \sqrt{f(s)}\, d\mu(s).
\end{aligned}$$

In the situation of Theorem 1 we may assume that $\varepsilon < 1/2n$, since we want to show that $\varepsilon \ge 1/2n - c/n^{5/4}$ for some $c > 0$. Define measures $\tilde{\lambda}$ and λ on T by $d\tilde{\lambda}(t) = \sqrt{f(t)}\, d\mu(t)$, $\lambda = \tilde{\lambda}/\tilde{\lambda}(T)$. Thus λ is a probability measure on T. We claim that

$$(10) \qquad \tilde{\lambda}(T) \ge (1 + \sqrt{2\varepsilon}/(1 - \sqrt{2n\varepsilon}))^{-1} =: \delta_2(\varepsilon, n)^{-1},$$

so that with $\delta(\varepsilon, n) := \delta_1(\varepsilon, n)\delta_2(\varepsilon, n)^2$,

$$(9'') \qquad \delta(\varepsilon, n) \ge \int_T \int_T \left(\frac{\alpha}{n} |h(s, t)| - 1 \right)^2 d\lambda(t)\, d\lambda(s) =: I$$

as in (9). To estimate I from below (in both (9) and (9″)), we shall use only the fact that ν and λ are probability measures.

Step 3. To prove (10), it suffices to show that

$$(11) \qquad \|f\|_{L_2(T, \mu)} \le \delta_2(\varepsilon, n).$$

Indeed, Hölder's inequality yields

$$1 = \|f\|^3_{L_1(T, \mu)} \leq \|f\|^2_{L_2(T, \mu)} \|f\|_{L_{1/2}(T, \mu)}.$$

Thus,

$$\tilde{\lambda}(T) = \int_T \sqrt{f(t)}\, d\mu(t) = \|f\|^{1/2}_{L_{1/2}(T, \mu)} \geq \|f\|^{-1}_{L_2(T, \mu)},$$

completing the proof of (10).

We first show that v, considered as a map on $X_2 \subset L_2(T, \mu)$, differs not too much from Id_X. Let $j: L_2(T, \mu) \to L_1(T, \mu)$ denote the inclusion map. Then $\tilde{v} := I_2 w j : L_2(T, \mu) \to L_2(T, \mu)$ actually maps X_2 into itself. We restrict our attention to $\tilde{v}: X_2 \to X_2$. Since for an operator S acting in a Hilbert space,

$$\mathrm{tr}(SS^*) = \mathrm{hs}(S)^2 = \pi_2(S)^2,$$

where $\mathrm{hs}(S)$ is the Hilbert–Schmidt norm of S, and since $\pi_2(I_2) = 1$ (cf., e.g., [TJ, §10]), we have

$$\begin{aligned} \pi_2(\mathrm{Id} - \tilde{v} : X_2 \to X_2)^2 &= \mathrm{tr}((\mathrm{Id} - \tilde{v})(\mathrm{Id} - \tilde{v}^*) : X_2 \to X_2) \\ &= n - 2\,\mathrm{tr}(\tilde{v} : X_2 \to X_2) + \mathrm{tr}(\tilde{v}\tilde{v}^* : X_2 \to X_2) \\ &\leq n - 2\,\mathrm{tr}(v) + \pi_2(\tilde{v} : X_2 \to X_2)^2 \\ &\leq n - 2n(1-\varepsilon) + (\pi_2(I_2)\|w\|)^2 = 2n\varepsilon. \end{aligned} \tag{12}$$

Hence

$$\|\mathrm{Id} - \tilde{v} : X_2 \to X_2\| \leq \sqrt{2n\varepsilon} < 1.$$

Thus $\tilde{v}$ is invertible on X_2 and

$$\|\tilde{v}^{-1} : X_2 \to X_2\| \leq 1/(1 - \sqrt{2n\varepsilon}). \tag{13}$$

To estimate $\|f\|_{L_2(T, \mu)}$ write

$$\begin{aligned} \|f\|_{L_2(T, \mu)} &= \left(\int_T \sum_{j=1}^n |f_j(s)|^2\, d\mu(s) \right)^{1/2} \\ &\leq \left(\int_T \sum_{j=1}^n |\tilde{v} f_j(s)|^2\, d\mu(s) \right)^{1/2} + \left(\int_T \sum_{j=1}^n |(\mathrm{Id} - \overline{v}) f_j(s)|^2\, d\mu(s) \right)^{1/2}. \end{aligned}$$

To estimate the first term, we use (6), $(4')$, and (7), to get the following inequality valid almost everywhere on T:

$$\begin{aligned}\sum_{j=1}^{n}|\overline{v}(f_j)(s)|^2 &\le \left\|\sum_{j=1}^{n}\overline{\overline{v}(f_j)(s)}\overline{v}(f_j)\right\|_\infty \\ &\le \sqrt{n}\int_T\left|\sum_{j=1}^{n}\overline{\overline{v}(f_j)(s)}f_j(t)\right|\,d\mu(t) \\ &= (\sqrt{n}/\sqrt{n})\int_T\left|\sum_{j=1}^{n}\overline{\overline{v}(f_j)(s)}h_j(t)\right|\,d\nu(t) \\ &\le \left(\int_T\left|\sum_{j=1}^{n}\overline{\overline{v}(f_j)(s)}h_j(t)\right|^2\,d\nu(t)\right)^{1/2} \\ &= \left(\sum_{j=1}^{n}|\overline{v}(f_j)(s)|^2\right)^{1/2}.\end{aligned}$$

Thus $\sum_{j=1}^n|\overline{v}(f_j)(s)|^2 \le 1$ for almost all s. To estimate the second term, observe that algebraically, the spaces X_∞, X_2, and X_1 coincide and $\overline{v}$ is algebraically the same map as $\tilde{v}$. Using the definition of the π_2-norm we get

$$\begin{aligned}\left(\int_T\sum_{j=1}^{n}|(\mathrm{Id}-\overline{v})f_j(s)|^2\,d\mu(s)\right)^{1/2} &= \left(\sum_{j=1}^{n}\|(\mathrm{Id}-\tilde{v})f_j\|^2\right)^{1/2} \\ &\le \pi_2(\mathrm{Id}-\tilde{v}:X_2\to X_2)\sup_{\|\eta\|_2=1}\left(\int_T\left|\sum_{j=1}^{n}\eta_j f_j(s)\right|^2\,d\mu(s)\right)^{1/2}.\end{aligned}$$

Combining these estimates with (12) and (7) we get

$$\tag{14}\begin{aligned}\|f\|_{L^2(T,\mu)} &\le 1+\frac{\sqrt{2\varepsilon n}}{\sqrt{n}}\sup_{\|\eta\|_2=1}\left(\int_T\left|\sum_{j=1}^{n}\eta_j h_j(s)\right|^2\,d\nu(s)\right)^{1/2}\|f\|_\infty^{1/2} \\ &= 1+\sqrt{2\varepsilon\|f\|_\infty}.\end{aligned}$$

To estimate $\|f\|_\infty$, we use the fact that, by (13), $\tilde{v}^{-1}:X_2\to X_2$ is well defined, and thus $\overline{v}^{-1}:X_1\to X_1$ is well defined too. Hence, by (6) and $(4')$, we have,

μ-a.e.,

$$\begin{aligned}
|f(s)|^2 &= \sum_{j=1}^{n} |f_j(s)|^2 \le \left\| \sum_{j=1}^{n} \overline{f_j(s)} \overline{v}\, \overline{v}^{-1} f_j \right\|_\infty \\
&\le \sqrt{n} \int_T \left| \sum_{j=1}^{n} \overline{f_j(s)} \overline{v}^{-1} f_j(t) \right| d\mu(t) \\
&\le \sqrt{n} \left(\int_T \left| \sum_{j=1}^{n} \overline{f_j(s)} \tilde{v}^{-1} f_j(t) \right|^2 d\mu(t) \right)^{1/2} \\
&\le \sqrt{n} \|\tilde{v}^{-1} : X_2 \to X_2\| \left(\int_T \left| \sum_{j=1}^{n} \overline{f_j(s)} f_j(t) \right|^2 d\mu(t) \right)^{1/2} \\
&\le \sqrt{n} \|\tilde{v}^{-1}\| (\|f\|_\infty^{1/2}/\sqrt{n}) \left(\int_T \left| \sum_{j=1}^{n} \overline{f_j(s)} h_j(t) \right|^2 d\nu(t) \right)^{1/2} \\
&= \|\tilde{v}^{-1}\| \cdot \|f\|_\infty^{3/2}.
\end{aligned}$$

This yields $\|f\|_\infty^{1/2} \le \|\tilde{v}^{-1}\|$ and thus, by (13) and (14),

$$\|f\|_{L_2(T,\mu)} \le 1 + \sqrt{2\varepsilon}/(1 - \sqrt{2n\varepsilon}) = \delta_2(\varepsilon, n).$$

We thus proved (11), (10), and $(9'')$.

Step 4. We now turn to estimating the double integral

$$I := \int_T \int_T \left(\frac{\alpha}{n} |h(s,t)| - 1 \right)^2 d\lambda(s)\, d\lambda(t)$$

from below, for arbitrary probability measure λ on T. The following facts on ultraspherical and Jacobi polynomials can be found in Müller [M], Levenštein [Lev], and Kabatyanskii–Levenštein [KaL].

Let $S^{n-1}(\mathbf{K}) = \{x \in \mathbf{K}^n \mid |x|_2 = 1\}$. For all $k, n \in \mathbf{N}$ there are polynomials R_k^n of degree k and functions $w_{ki}^n : S^{n-1}(\mathbf{K}) \to \mathbf{K}$, $i = 1, \ldots, r_k^n$, such that for all $x, y \in S^{n-1}(\mathbf{K})$ the following "addition formula" for R_k^n holds:

$$R_k^n(|\langle x, y\rangle|^2) = \sum_{i=1}^{r_k^n} w_{ki}^n(x) \overline{w_{ki}^n(y)}. \tag{15}$$

These functions differ in the real and complex cases. In terms of Jacobi polynomials $P_k^{\alpha,\beta}$, which are orthogonal polynomials on $[-1, 1]$ with weight function $(1-v)^\alpha (1+v)^\beta$, one has

$$R_k^n(v) = c_{nk} P_k^{(n-3)/2, -1/2}(2v - 1), \qquad \mathbf{K} = \mathbf{R} \text{ and } v \in [0, 1],$$
$$R_k^n(v) = c_{nk} P_k^{n-2, 0}(2v - 1), \qquad \mathbf{K} = \mathbf{C} \text{ and } v \in [0, 1].$$

For $k = 0, 1, 2$ we have, with a suitable normalization,

$$\begin{aligned} &\left.\begin{aligned} R_0^n(v) &= 1 \\ R_1^n(v) &= nv - 1 \end{aligned}\right\}, && \mathbf{K} = \mathbf{R} \text{ or } \mathbf{C}, \\ &R_2^n(v) = (n+2)(n+4)v^2 - 6(n+2)v + 3, && \mathbf{K} = \mathbf{R}, \\ &R_2^n(v) = (n+1)(n+2)v^2 - 4(n+1)v + 2, && \mathbf{K} = \mathbf{C}. \end{aligned} \tag{16}$$

The functions w_{ki}^n are harmonic polynomials which are homogeneous of degree $2k$ in x in the real case and homogeneous of degrees k in x and $\overline{x}$ in the complex case; r_k^n are the dimensions of these spaces of polynomials,

$$r_k^n = \binom{n+2k-1}{n-1} - \binom{n+2k-3}{n-1}, \qquad \mathbf{K} = \mathbf{R},$$

$$r_k^n = \binom{n+k-1}{n-1}^2 - \binom{n+k-2}{n-1}^2, \qquad \mathbf{K} = \mathbf{C},$$

but we will not use the last facts. The use of (15) to estimate I was inspired by Levenštein's bounds for the number of balls in spherical packing problems.

For $s \in T$, let $x_s := (1/\sqrt{n})(h_j(s))_{j=1}^n \in \mathbf{K}^n$. By (7), $|x_s|_2 = 1$ for almost all s. Since $|\langle x_s, x_t\rangle| \le 1$, we find

$$\left(\frac{\alpha}{n}|h(s,t)| - 1\right)^2 = (\alpha|\langle x_s, x_t\rangle| - 1)^2 \ge \frac{1}{(\alpha+1)^2}(\alpha^2|\langle x_s, x_t\rangle|^2 - 1)^2.$$

Set $F(v) := (\alpha^2 v - 1)^2/(\alpha+1)^2$, for $v \in [0, 1]$. Then double integral I is bounded from below by

$$I \ge \int_T \int_T F(|\langle x_s, x_t\rangle|^2)\, d\lambda(s)\, d\lambda(t). \tag{17}$$

An easy calculation shows, using (16), that

$$F(v) = f_0^n + f_1^n R_1^n(v) + f_2^n R_2^n(v),$$

where

$$\begin{aligned} f_0^n &= \frac{1}{(\alpha+1)^2}\left(1 - \frac{2\alpha^2}{n} + \frac{3\alpha^4}{n(n+2)}\right), \\ f_1^n &= \frac{1}{(\alpha+1)^2}\left(\frac{6\alpha^4}{n(n+4)} - \frac{2\alpha^2}{n}\right), \\ f_2^n &= \frac{1}{(\alpha+1)^2}\frac{\alpha^4}{(n+2)(n+4)} \quad \text{if } \mathbf{K} = \mathbf{R}; \\ f_0^n &= \frac{1}{(\alpha+1)^2}\left(1 - \frac{2\alpha^2}{n} + \frac{2\alpha^4}{n(n+1)}\right), \\ f_1^n &= \frac{1}{(\alpha+1)^2}\left(\frac{4\alpha^4}{n(n+2)} - \frac{2\alpha^2}{n}\right), \\ f_2^n &= \frac{1}{(\alpha+1)^2}\frac{\alpha^4}{(n+1)(n+2)} \quad \text{if } \mathbf{K} = \mathbf{C}. \end{aligned} \tag{18}$$

Hence $f_k^n \geq 0$ if $\alpha \geq \sqrt{(n+4)/3}$ if $\mathbf{K}=\mathbf{R}$ or $\alpha \geq \sqrt{(n+2)/2}$ if $\mathbf{K}=\mathbf{C}$ for $k=0, 1, 2$. Using this, (15) and (17), and the fact that λ is a probability measure on T, we get

$$
(19) \qquad \begin{aligned} I &\geq f_0^n + \sum_{k=1}^{2} f_k^n \int_T \int_T R_k^n(|\langle x_s, x_t\rangle|^2)\, d\lambda(s)\, d\lambda(t) \\ &= f_0^n + \sum_{k=1}^{2} f_k^n \sum_{i=1}^{r_k} \left| \int_T w_{ki}^n(x_s)\, d\lambda(s)\right|^2 \geq f_0^n . \end{aligned}
$$

Now we are prepared for short proofs of our results. Roughly speaking, they are based on the fact that for $\alpha \sim \sqrt{n}$, f_0^n is of order $1/n$. More delicate calculations show much more precise estimates.

Proof of Theorem 1. Let $\lambda(X) = \sqrt{n}(1-\varepsilon)$. If $\varepsilon \geq 1/(2n)$, the conclusion is obviously satisfied. Assume that $\varepsilon < 1/(2n)$. By (9″), (18), and (19) we get

$$\delta(\varepsilon, n) = 2\varepsilon\left(1 + \frac{\sqrt{2\varepsilon}}{1-\sqrt{2\varepsilon n}}\right)^2 \geq I \geq f_0^n :$$

Putting $c_n := \sqrt{f_0^n}$ and $\delta = \sqrt{2\varepsilon}$, this means

$$\delta^2 - \frac{1+c_n\sqrt{n}}{\sqrt{n}-1}\delta + \frac{c_n}{\sqrt{n}-1} \leq 0$$

and thus

$$\sqrt{2\varepsilon} = \delta \geq \frac{1}{2(\sqrt{n}-1)}\{(1+c_n\sqrt{n}) - \sqrt{(1-c_n\sqrt{n})^2 + 4c_n}\} =: \delta_n .$$

Thus $\varepsilon \geq c/n$ for some $c > 0$. Asymptotically, in the real case

$$1 - c_n\sqrt{n} = (\sqrt{2}-1) + O(1/\sqrt{n}), \qquad \delta_n = 1/\sqrt{n} - O(1/n),$$

and hence

$$\varepsilon \geq 1/2n - O(1/n^{3/2}).$$

In the complex case the estimate is slightly worse; namely,

$$1 - c_n\sqrt{n} = O(1/\sqrt{n}), \qquad \delta_n = 1/\sqrt{n} - O(1/n^{3/4}),$$

and hence

$$\varepsilon \geq 1/2n - O(1/n^{5/4}).$$

In both cases we get the bound

$$\lambda(X) \leq \sqrt{n}(1-\varepsilon) \leq \sqrt{n} - 1/2\sqrt{n} + O(1/n^{3/4}).$$

This completes the proof. □

In the case of 2-dimensional real spaces, the choice of $\alpha = \frac{3}{2}$ leads to the numerical bound $\lambda(X_2^{\mathbf{R}}) \leq 1.378$ given in Remark 1.

Observe that Proposition 2 follows directly from Proposition 3.

Proof of Proposition 3. Let $\pi_1(X) = \sqrt{n}/(1-\varepsilon)$. By Step 1, (9), and (19) we get

$$2\alpha/\sqrt{n} + (\alpha/\sqrt{n} - 1)^2 \geq f_0^n .$$

Let $\alpha = \sqrt{n+2}$ if $\mathbf{K} = \mathbf{R}$ and $\alpha = \sqrt{n+1}$ if $\mathbf{K} = \mathbf{C}$. A calculation using formula (18) for f_0^n implies a lower bound for ε which turns out to yield

$$\pi_1(X^{\mathbf{R}}) \geq n/f(n, n(n+1)/2), \qquad \pi_1(X^{\mathbf{C}}) \geq n/f(n, n^2),$$

where $f(n, N)$ is the function given in (3). □

Proof of Remark 3. We indicate the necessary changes in Step 1 of the previous argument to get a lower bound for $\pi_p(X)$ if $1 < p < 2$. Taking a p-summing factorization of Id_X, one finds again a compact space T and a probability measure μ on T and the isometric image $X_\infty \subset L_\infty(T, \mu)$ of X such that

$$\|g\|_\infty \leq \pi_p(X) \left(\int_T |g(s)|^p \, d\mu(s) \right)^{1/p} . \tag{4''}$$

In $X_p \subset L_p(T, \mu)$ there is a basis $(f_j)_{j=1}^n$ such that

$$n \int_T f_j(s) \overline{f_k(s)} f(s)^{p-2} \, d\mu(s) = \delta_{jk} , \tag{5'}$$

where $f = (\sum_{j=1}^n |f_j|^2)^{1/2}$ [Lew1]. Let $d\nu(s) = f(s)^p \, d\mu(s)$, $h_j(s) := \sqrt{n} f_j(s)/f(s)$, and $h(s, t) := \sum_{j=1}^n h_j(s)\overline{h_j(t)}$. Then the orthogonality relations (7) hold and the previous arguments show that

$$\begin{aligned} n &\leq \pi_p(X) \int_T \left(\int_T \left| \sum_{j=1}^n n f_j(s) \overline{f_j(t)} \right|^p d\mu(t) \right)^{1/p} d\mu(s) \\ &\leq \pi_p(X) \left(\int_T \int_T |h(s, t)|^p \, d\nu(t) \, d\nu(s) \right)^{1/p} = \pi_p(X) \|h\|_p . \end{aligned}$$

By (7), $\|h\|_2 = \sqrt{n}$; hence Hölder's inequality shows that

$$\|h\|_p \leq \|h\|_1^{2/p-1} \|h\|_2^{2/p'} = n^{1/p'} \|h\|_1^{2/p-1} .$$

Define $\varepsilon \geq 0$ by $\pi_p(X) = \sqrt{n}/(1-\varepsilon)$. We get

$$\int_T \int_T |h(s, t)| \, dv(s) \, dv(t) \geq \left(\frac{n^{1/p}}{\pi_p(X)} \right)^{p/(2-p)} = (1-\varepsilon)^{p/(2-p)} \sqrt{n} .$$

As in (9), this yields (with $\alpha = \sqrt{n}$)

$$\begin{aligned} I &= \int_T \int_T \left(\frac{1}{\sqrt{n}} |h(s, t)| - 1 \right)^2 d\nu(s) \, d\nu(t) \\ &\leq 2(1 - (1-\varepsilon)^{p/(2-p)}) \leq (1/p - \tfrac{1}{2})^{-1} \varepsilon . \end{aligned}$$

Since $I \geq f_0^n$ and $f_0^n = 2/n - O(1/n^{3/2})$ if $\mathbf{K} = \mathbf{R}$ and $f_0^n = 1/n - O(1/n^{3/2})$ if $\mathbf{K} = \mathbf{C}$, we conclude

$$\pi_p(X) \geq \sqrt{n} + (2/p - 1)/\sqrt{n} - O(1/n) \quad \text{if } \mathbf{K} = \mathbf{R},$$
$$\pi_p(X) \geq \sqrt{n} + (1/p - \tfrac{1}{2})/\sqrt{n} - O(1/n) \quad \text{if } \mathbf{K} = \mathbf{C}. \quad \square$$

3. Spaces with nearly extremal λ-, π_1-, and gl-constants

We now construct a class of spaces with very large projection constants and very small 1-summing constants. The spaces are of a concrete nature: they are translation-invariant subspaces of l_∞^N, the elements of l_∞^N being considered as functions on a finite group. They may also be constructed to have Gordon–Lewis constants of maximal order and extremal Euclidean distance and Sidon constants, while being character spaces over Λ_4-sets. The estimates for λ and π_1 are based on the main idea of [Kö], which can be formulated in the following lemma. In the formulae below, $\|\cdot\|_2$ denotes the usual l_2-norm on $\mathbf{K}^n$ and $\mathbf{K}^N$.

Lemma 6. *Let A be a scalar $N \times n$ matrix of rank n and $N > n$. We denote the N rows of A by $x_s \in \mathbf{K}^n$ and the n columns of A by $y_j \in \mathbf{K}^N$ and let $X_\infty^n := \operatorname{Span}[y_1, \dots, y_n] \subseteq l_\infty^N$, equipped with the l_∞^N-norm. If for some $0 < \alpha < 1$*

$$\|x_s\|_2 = \sqrt{n} \quad \textit{and} \quad |\langle x_s, x_t\rangle| \leq \alpha n \quad \textit{for all } s \neq t,\ 1 \leq s,\ t \leq N, \tag{20}$$

then $\dim X_\infty^n = n$ and

$$\lambda(X_\infty^n) \geq (1 - n/N)(1/\alpha) + n/N. \tag{21}$$

If additionally

$$\|y_j\|_2 = \sqrt{N} \quad \textit{and} \quad \langle y_j, y_k\rangle = 0 \quad \textit{for all } j \neq k,\ 1 \leq j,\ k \leq n, \tag{22}$$

then

$$\pi_1(X_\infty^n) \leq Nn\alpha/(N - n + \alpha n). \tag{23}$$

If α satisfies $\alpha = \sqrt{(N-n)/n(N-1)}$, equality holds in (21) and (23) and $\lambda(X_\infty^n)\pi_1(X_\infty^n) = n$.

Remark 5. The vectors x_s form *equiangular lines* if for all $s \neq t$, $|\langle x_s, x_t\rangle| = \alpha n$ holds. For $\alpha = \sqrt{(N-n)/(n(N-1))}$ the inequality in (20) actually implies the equality $|\langle x_s, x_t\rangle| = \alpha n$.

Clearly, the norm of a vector $\sum_{j=1}^n \lambda_j y_j \in X_\infty^n$ is just given by

$$\left\| \sum_{j=1}^n \lambda_j y_j \right\|_\infty = \sup_{1 \leq s \leq N} |\langle \lambda, x_s\rangle|,$$

where $\lambda = (\lambda_j)_{j=1}^n$ and $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathbf{K}^n$. An application of the Hahn–Banach theorem shows that the dual space $(X_\infty^n)^*$ has the absolutely convex hull of the vectors $(x_s)_{s=1}^N \subset \mathbf{K}^n$ as its unit ball. Clearly, (22) implies that A has rank n, but (20) by itself does not necessarily imply it.

Proof. Let $e_j \in \mathbf{K}^n$ denote the standard unit vector basis. Since $y_j = Ae_j$, $AA^* = (\langle x_s , x_t \rangle)_{s,t=1}^N$ maps l_∞^N into X_∞^n. Let Id_N denote the $N \times N$ identity matrix. Then

$$B = (b_{st})_{s,t=1}^N := \frac{1}{(1-\alpha)n}(\langle x_s , x_t \rangle)_{s,t=1}^N - \mathrm{Id}_N : l_\infty^N \to l_\infty^N$$

maps X_∞^N into itself. One has

$$\operatorname{tr}(B|_{X_\infty^n}) = \frac{N}{1-\alpha} - n, \qquad |b_{st}| \le \frac{\alpha}{1-\alpha} \quad \text{for all } 1 \le s,\ t \le N.$$

The trace is always bounded by the nuclear norm (cf., e.g., [TJ]) and thus

$$|\operatorname{tr}(B|_{X_\infty^n})| \le \nu(B|_{X_\infty^n}) \le \lambda(X_\infty^n)\nu(B : l_\infty^N \to l_\infty^N).$$

On l_∞^N, the nuclear norm is $\nu(B) = \sum_{t=1}^n \sup_{1\le s\le N} |b_{st}| \le N\alpha/(1-\alpha)$ (cf., e.g., [TJ]). Hence

$$\lambda(X_\infty^n) \ge \frac{1}{\alpha} - \frac{n}{N}\frac{1-\alpha}{\alpha} = \left(1 - \frac{n}{N}\right)\frac{1}{\alpha} + \frac{n}{N}.$$

If additionally (22) is satisfied, i.e., $A^*A = N\,\mathrm{Id}_n$, then $P := (1/N)AA^*$ is a projection onto X_∞^N. Indeed, $\operatorname{tr}(P) = n$ and

$$Py_j = (1/N)(AA^*)Ae_j = Ae_j = y_j \quad \text{for } 1 \le j \le n.$$

Therefore

$$\lambda(X_\infty^n) \le \|P\| = \sup_{1\le s\le N} \sum_{t=1}^N \frac{1}{N}|\langle x_s , x_t \rangle| = \frac{n}{N}(1 + (N-1)\alpha).$$

For $\alpha = \sqrt{(N-n)/n(N-1)}$ this is equal to $(1-n/N)(1/\alpha)+n/N$ and equality holds in (21). Assuming (22), B is a multiple of the Id on X_∞^n,

$$B|_{X_\infty^n} = \left(\frac{N}{n}\frac{1}{1-\alpha} - 1\right)\mathrm{Id}_{X_\infty^n}.$$

Since the 1-summing norm is injective and bounded by the nuclear norm, we find

$$\left(\frac{N}{n}\frac{1}{1-\alpha} - 1\right)\pi_1(X_\infty^n) = \pi_1(B|_{X_\infty^n}) \le \nu(B) = \frac{N\alpha}{1-\alpha},$$

$$\pi_1(X_\infty^n) \le Nn\alpha/(N - n(1-\alpha)).$$

This shows (23). For $\alpha = \sqrt{(N-n)/n(N-1)}$, the trace-duality (1) and the formula for $\lambda(X_\infty^n)$ yield the equality in (23) since

$$n \le \lambda(X_\infty^n)\pi_1(X_\infty^n) \le \frac{N-n+n\alpha}{N\alpha} \cdot \frac{Nn\alpha}{N-n(1-\alpha)} = n,$$

completing the proof. □

Proof of Remark 2. The upper estimate follows from Proposition 2. If $n = 3$ the six diagonals of the dodecahedron in $\mathbf{R}^3$ yield six equiangular (normalized in l_2^3) vectors $x_s \in \mathbf{R}^3$ with $|\langle x_s, x_t\rangle| = 1/\sqrt{5}$, for $s \neq t$, $1 \leq s, t \leq 6$. By (21), the corresponding space X_∞^3 satisfies $\lambda(X_\infty^3) \geq (1+\sqrt{5})/2$. The dual unit ball of X_∞^3 is the icosahedron (= absolutely convex hull of the x_s's); hence the unit ball of X_∞^3 is the dodecahedron. For $n = 2$ take the three diagonals of the (self-dual) regular hexagon in $\mathbf{R}^2$. Condition (22) is satisfied in both cases and $\alpha = \sqrt{(N-n)/n(N-1)}$ holds, too. □

Proof of Proposition 4(b). The Kerdock code [Ke] (see also Levenštein [Lev]) shows that for $n = 4^m$ there exist $N = n^2/2$ real vectors $x_s \in \mathbf{R}^n$ with coordinates ± 1 and $\|x_s\|_2 = \sqrt{n}$, $|\langle x_s, x_t\rangle| \leq \sqrt{n}$ for $s \neq t$, $1 \leq s, t \leq N$. Lemma 6 shows that the corresponding space X_∞^n satisfies

$$\lambda(X_\infty^n) \geq \left(1 - \frac{n}{N}\right)\sqrt{n} + \frac{n}{N} = \sqrt{n} - \frac{2}{\sqrt{n}} + \frac{2}{n}.$$

On the other hand, by formula (3),

$$\lambda(X_\infty^n) \leq f(n, n^2/2) \leq \sqrt{n} - 1/\sqrt{n}. \quad \square$$

We proceed now to the main construction of n-dimensional translation-invariant subspaces of l_∞^N which have extremal properties stated in Propositions 4 and 5. Such a subspace will be defined by a sequence of nonnegative integers $d_1, \dots, d_n$, which are distinct $\bmod N$. Set

$$y_j := \left(\exp\left(\frac{2\pi i}{N} d_j s\right)\right)_{s=0}^{N-1} \in \mathbf{C}^n \quad \text{for } j = 1, \dots, n. \tag{24}$$

Then put $X_\infty^n := \mathrm{Span}[y_1, \dots, y_n] \subset l_\infty^N$, equipped with the l_∞-norm. Clearly, $\dim X_\infty^n = n$. The y_j's can be identified with characters on the cyclic group $\mathbf{Z}_N$. In fact, for $j = 1, \dots, n$ and $s \in \{0, \dots, N-1\} = \mathbf{Z}_N$,

$$y_j(s) = \exp((2\pi i/N) d_j s). \tag{25}$$

In particular, X_∞^n is a translation-invariant space.

The properties of X_∞^n depend on number theoretic properties of the defining sequence $d_1, \dots, d_n$. In particular, we shall use some facts about so-called finite B_2-sequences, which can be found, e.g., in [HR].

Theorem 7. *Let $m \in \mathbf{N}$, p be a prime number, and $q = p^m$.*

(a) *Let $M = q^2 - 1$. Then there exist q integers $0 \leq d_1, \dots, d_q < M$ such that the $q(q-1) = M - (q-1)$ differences $d_i - d_j$, $i \neq j$, when reduced* $\bmod M$, *are all the positive integers less than M which are not divisible by $(q+1)$.*

(b) *Let $N = q^2+q+1$. Then there exist $(q+1)$ integers $0 \leq d_0, \dots, d_q < N$ such that the $q(q+1) = N-1$ differences $d_i - d_j$, $i \neq j$, when reduced* $\bmod N$, *are all the positive integers less than N.*

Now, let $m \in \mathbf{N}$, p be a prime number, $q = p^m$. Let $n := q+1$ and $N := q^2 + q + 1 = n^2 - n + 1$. Let $d_0, d_1, \dots, d_q$ be the sequence from Theorem 7(b) and let y_j be defined by (24), for $j = 0, \dots, q$. The space X_∞^n satisfies the requirement of Proposition 4(a), i.e.,

$$\lambda(X_\infty^n) = f(n, N) \quad \text{and} \quad \pi_1(X_\infty^n) = n/f(n, N).$$

Proof of Proposition 4(a). Let $\alpha = \sqrt{(N-n)/n(N-1)} = \sqrt{(n-1)/n}$. Then $(1 - n/N)(1/\alpha) + n/N = f(n, N)$. By Lemma 6, it suffices to check (20) and (22) for the matrix A built with column vectors y_j. Clearly $\|y_j\|_2 = \sqrt{N}$ and

$$\langle y_j, y_k \rangle = \frac{1}{N} \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N}(d_j - d_k)s\right) = 0 \quad \text{for } j \neq k, \ 0 \leq j, \ k \leq q.$$

For the row vectors x_s we have $\|x_s\|_2 = \sqrt{n}$, for $s = 0, \dots, N-1$. Moreover, for $s \neq t$, $0 \leq s, t \leq N-1$ we have, with $\theta = s - t$,

$$\begin{aligned} |\langle x_s, x_t \rangle|^2 &= \left| \sum_{j=0}^{q} \exp\left(\frac{2\pi i}{N} d_j \theta\right) \right|^2 = \sum_{j,k=0}^{q} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right) \\ &= \left\{ \sum_{j=k} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right) + \sum_{j \neq k} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right) \right\} \\ &= n + \sum_{l=1}^{N-1} \exp\left(\frac{2n i}{N} l\theta\right) = n - 1, \end{aligned}$$

since $\sum_{l=0}^{N-1} \exp((2\pi i/N)l\theta) = 0$. Hence $|\langle x_s, x_t \rangle| = \alpha n$ for $s \neq t$, completing the proof. □

Note that since X_∞^n is translation invariant a general averaging argument shows that $\lambda(X_\infty^n)$ is equal to the norm of the orthogonal projection from l_∞^N onto X_∞^n. Since the projection P constructed in the proof of Lemma 6 is orthogonal, for translation-invariant spaces the first part of the proof of Lemma 6 can be omitted.

The proof above shows that the x_s's form equiangular vectors in $\mathbf{C}^n$. In fact, by Levenštein [Lev], for this $\alpha = \sqrt{n-1}/n$, the number N is maximal. This example also shows that the lower estimate of the double integral in Step 4 of the previous section and in the proof of Proposition 3 is almost the best possible, in general (up to terms of $O(n^{3/2})$).

Proposition 4 is a slight improvement of the result from [Kö], where a similar construction was considered, with $N = q^2$ and $d_j = j^2$, $j = 1, \dots, q = n$. This gave X_∞^n such that $\lambda(X_\infty^n) = \sqrt{n} - 1/\sqrt{n} + O(1/n)$, while here we have $\lambda(X_\infty^n) = \sqrt{n} - 1/2\sqrt{n} + O(1/n)$.

We pass now to the main construction of this section, which was announced in Proposition 5. Let $m \in \mathbf{N}$ and p be an odd prime number. Set $n := p^m$, $M := n^2 - 1$, and $N := M/2 = (n-1)(n+1)/2$. Let $0 \leq d_1, \dots, d_n < M$ be the sequence from Theorem 7(a) such that the differences $d_i - d_j$, for $i \neq j$, $1 \leq i, j \leq n$, when reduced mod M, are all the positive integers less than M which are not divisible by $(n+1)$.

Let $y_j \in \mathbf{C}^N$ $(j = 1, \dots, n)$ be defined by (24), i.e.,

$$y_j := (\exp((2\pi i/N)d_j s))_{s=0}^{N-1} \in \mathbf{C}^N \quad \text{for } j = 1, \dots, n.$$

Let $X_\infty^n := \mathrm{Span}[y_1, \dots, y_n] \subset l_\infty^N$.

Let $\|\cdot\|_p$ denote the L_p-form on $\mathbf{C}^N$ in the function normalization, for $1 \leq p < \infty$, i.e., for $y = (y(s))_{s=0}^{N-1} \in \mathbf{C}^N$, $\|y\|_p = ((1/N)\sum_{s=0}^{N-1} |y(s)|^p)^{1/p}$. Hence $\|y\|_p \leq \|y\|_q$ for $p \leq q$. Let $L_p^N = (\mathbf{C}^N, \|\cdot\|_p)$ and $X_p^n := \mathrm{Span}[y_1, \dots, y_n] \subset L_p^N$, with the inherited L_p-norm.

Now we are ready to show that the space X_∞^n satisfies the conclusions of Proposition 5.

Proof of Proposition 5. (a), (b) Let $\alpha = 1/\sqrt{n}$. Let $x_s \in \mathbf{C}^n$ $(s = 0, \dots, N-1)$ be the row vectors of the $N \times n$ matrix A built from column vectors $y_1, \dots, y_n$. We check that (20) and (22) of Lemma 6 hold. Clearly, $\|y\|_2 = \sqrt{N}$ and for $j \neq k$, $1 \leq j, k \leq n$,

$$\langle y_j, y_k \rangle = \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N}(d_j - d_k)s\right) = 0.$$

The vectors $(x_s)_{s=0}^{N-1} \in l_2^n$ are almost equiangular; for $s \neq t$, $0 \leq s, t \leq N-1$ we have with $\theta := s - t$,

$$\begin{aligned}
|\langle x_s, x_t \rangle|^2 &= \left|\sum_{j=1}^{n} \exp\left(\frac{2\pi i}{N} d_j \theta\right)\right|^2 = \sum_{j,k=1}^{n} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right) \\
&= \left\{\sum_{j=k} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right) + \sum_{j \neq k} \exp\left(\frac{2\pi i}{N}(d_j - d_k)\theta\right)\right\} \\
&= n + \sum_{l=1}^{M-1} \exp\left(\frac{2\pi i}{N} l\theta\right) - \sum_{m=1}^{n-2} \exp\left(\frac{2\pi i}{N} m(n+1)\theta\right) \\
&= n - 1 - \sum_{m=1}^{n-2} \exp\left(\frac{2\pi i}{n-1} 2\theta m\right).
\end{aligned}$$

If $\theta = s - t$ is not divisible by $(n-1)$, the last sum is equal to -1; otherwise it is $(n-2)$. Therefore

$$(26) \qquad |\langle x_s, x_t \rangle| = \begin{cases} \sqrt{n}, & (s-t) \pmod{M} \neq j(n-1),\ j = 1, \dots, n, \\ 1, & (s-t) \pmod{M} = j(n-1),\ j = 1, \dots, n. \end{cases}$$

Thus Lemma 6 yields

$$\lambda(X_\infty^n) \geq \left(1 - \frac{n}{N}\right)\frac{1}{\alpha} + \frac{n}{N} = \sqrt{n} - \frac{2}{\sqrt{n}} + O\left(\frac{1}{n}\right),$$

$$\pi_1(X_\infty^n) \leq \frac{Nn\alpha}{N - n + n\alpha} = \sqrt{n} + \frac{2}{\sqrt{n}} - O\left(\frac{1}{n}\right).$$

Since $P := (1/N)(\langle x_s, x_t\rangle)_{s,t=0}^{N-1} : l_\infty^N \to l_\infty^N$ is a projection onto X_∞^n, using (26) we find that

$$\begin{aligned}\lambda(X_\infty^n) \leq \|P\| &= \sup_{0\leq s\leq N-1} \sum_{t=0}^{N-1} \frac{1}{N}|\langle x_s, x_t\rangle| \\ &= \frac{n}{N}\left(1 + (N - (n+1))\frac{1}{\sqrt{n}} + n\cdot\frac{1}{n}\right) = \sqrt{n} - \frac{2}{\sqrt{n}} + O\left(\frac{1}{n}\right).\end{aligned} \tag{27}$$

By trace-duality,

$$\pi_1(X_\infty^n) \geq n/\lambda(X_\infty^n) = \sqrt{n} + 2/\sqrt{n} - O(1/n),$$

proving (a) and (b).

(c) For an arbitrary n-dimensional space X_n, $d(X_n, l_2^n) \leq \sqrt{n}$, and this estimate is given by the so-called John's ellipsoid of maximal volume, contained in the unit ball B_E (cf., e.g., [TJ, §§9, 15]). Therefore we need to show that $d(X_\infty^n, l_2^n) \geq \sqrt{n}$. Clearly X_2^n and l_2^n are isometric. Let $\beta_j := \exp((2\pi i/N)d_j)$ for $j = 1, \ldots, n$. Then $I : X_\infty^n \to X_\infty^n$ defined by $\sum_{j=1}^n a_j y_j \to \sum_{j=1}^n \beta_j a_j y_j$ is an isometry and $I^N = \mathrm{Id}_{X_\infty^n}$. Any inner product $[\cdot, \cdot]$ on X_∞^n invariant with respect to I is diagonal in the basis (y_j). Indeed, if $[x, z] = \sum_{j,k=1}^n t_{jk} a_j \overline{b_k}$, for $x = \sum_{j=1}^n a_j y_j$ and $z = \sum_{k=1}^n b_k y_k$, then the condition $[Ix, Iz] = [x, z]$ implies that $t_{jk} = t_{jk}\beta_j\overline{\beta}_k$. For $j \neq k$, however,

$$\beta_j\overline{\beta}_k = \exp((2\pi i/N)(d_j - d_k)) \neq 1$$

and thus $t_{jk} = 0$. Note here that $d_j - d_k \neq N \pmod N$ since N is divisible by $(n+1)$. Now let $(\cdot, \cdot)$ be an inner product on X_∞^n which determines the distance $d(X_\infty^n, l_2^n) = d$, normalized so that

$$(1/d^2)\|x\|_\infty^2 \leq (x, x) \leq \|x\|_\infty^2 \quad \text{for } x \in X_\infty^n.$$

Define the inner product $[\cdot, \cdot]$ on X_∞^n by

$$[x, z] = \frac{1}{N}\sum_{k=0}^{N-1}(I^k x, I^k z) \quad \text{for } x, z \in X_\infty^n.$$

Clearly,

$$(1/d^2)\|x\|_\infty^2 \leq [x, x] \leq \|x\|_\infty^2 \quad \text{for } x \in X_\infty^n.$$

Moreover, $[x, z] = [Ix, Iz]$ for $x, z \in X_\infty^n$. Therefore there exist positive numbers $\lambda_1, \ldots, \lambda_n$ such that

$$[x, z] = \sum_{j=1}^n \lambda_j a_j \overline{b_j}$$

for $x = \sum_{j=1}^n a_j y_j$ and $z = \sum_{j=1}^n b_j y_j$. Since $[x, x] \le \|x\|_\infty^2$, then $\lambda_j \le 1$ for $j = 1, \dots, n$.

Set $z := \sum_{j=1}^n y_j$. Then $[z, z] = \sum_{j=1}^n \lambda_j$. Taking $s = 0$, we find

$$\|z\|_\infty = \left\| \sum_{j=1}^n y_j \right\|_\infty = \sup_{0 \le s \le N-1} \left| \sum_{j=1}^n \exp\left(\frac{2\pi i}{N} d_j s\right) \right| = n. \tag{28}$$

So

$$d \ge \sup_{x \ne 0} \|x\|_\infty / [x, x]^{1/2} \ge \frac{n}{(\sum_{j=1}^n \lambda_j)^{1/2}} \ge \sqrt{n},$$

since $\lambda_j \le 1$ for $j = 1, \dots, n$. Thus $d(X_\infty^n, l_2^n) = d(X_\infty^n, X_2^n) = d = \sqrt{n}$. Also, $\lambda_j = 1$ for $j = 1, \dots, n$; hence the distance $d(X_\infty^n, l_2^n)$ is given by the inclusion map $I_{\infty 2} : X_\infty^n \to X_2^n$. In fact, the natural inner product is the only inner product diagonal with respect to the character basis (y_j), which determines the distance.

(d) Set $z := \sum_{j=1}^n y_j$. By (28), $\|z\|_\infty = n$. Set $\varepsilon_j := \exp((2\pi i/M) d_j)$, for $j = 1, \dots, n$ and $w := \sum_{j=1}^n \varepsilon_j y_j$. Then

$$\begin{aligned}
\|w\|_\infty^2 &= \sup_{0 \le s \le N-1} \left| \sum_{j=1}^n \exp\left(\frac{2\pi i}{M} d_j (2s+1)\right) \right|^2 \\
&= \sup_{0 \le s \le N-1} \sum_{j,k=1}^n \exp\left(\frac{2\pi i}{M} (d_j - d_k)(2s+1)\right) \\
&= \sup_{0 \le s \le N-1} \left[n + \sum_{j \ne k} \exp\left(\frac{2\pi i}{M} (d_j - d_k)(2s+1)\right) \right] \\
&= n.
\end{aligned}$$

For the last equality observe that since $(2\pi i/M)(n+1) = 2\pi i/(n-1)$, the latter sum, for every $0 \le s \le N-1$, is equal to

$$\sum_{l=1}^{m-1} \exp\left(\frac{2\pi i}{M} l(2s+1)\right) - \sum_{m=1}^{n-2} \exp\left(\frac{2\pi i}{M} m(n+1)(2s+1)\right) = -1 - (-1) = 0.$$

Thus $\|w\|_\infty = \sqrt{n}$. Therefore $\mathrm{ubc}(y_j) \ge \|z\|_\infty / \|w\|_\infty = \sqrt{n}$. On the other hand, an arbitrary character basis (y_j) in a translation-invariant space satisfies $\mathrm{ubc}(y_j) \le \sqrt{n}$. Indeed, observe that the uniqueness of the John's ellipsoid of maximal volume (cf. [TJ, §15]) implies that the translation operator I, defined in (c), is an isometry in the Hilbert space H associated to this ellipsoid. This yields that (y_j) form an orthogonal, hence also 1-unconditional, basis in H. So $\mathrm{ubc}(y_j \mid X_\infty^n) \le \sqrt{n}\, \mathrm{ubc}(y_j \mid H) = \sqrt{n}$.

(e) The vectors $(y_j)_{j=1}^n$ form the natural character basis in the translation-invariant space X_∞^n (cf. [25]) and so the Sidon constant $S(X_\infty^n)$ is the smallest constant $c > 0$ such that, for all $(a_j)_{j=1}^n \in \mathbf{C}^n$,

$$\sum_{j=1}^n |a_j| \le c \left\| \sum_{j=1}^n a_j y_j \right\|_\infty .$$

Since the left-hand side norm is 1-unconditional, it easily follows by the triangle inequality that $\operatorname{ubc}(y_j) \le S(X_\infty^n)$. On the other hand, we have

$$\begin{aligned} \sum_{j=1}^n |a_j| &= \sup_{0 \le s \le N-1} \sup_{|\eta_j|=1} \left| \sum_{j=1}^n \eta_j a_j \exp\left(\frac{2\pi i}{N} d_j s\right) \right| \\ &= \sup_{|\eta_j|=1} \left\| \sum_{j=1}^n \eta_j a_j y_j \right\|_\infty \le \operatorname{ubc}(y_j) \left\| \sum_{j=1}^n a_j y_j \right\|_\infty , \end{aligned}$$

and so $S(X_\infty^n) \le \operatorname{ubc}(y_j)$. Thus, by (d), $S(X_\infty^n) = \sqrt{n}$ and this is the maximal value for any Sidon constant.

(f) First observe that on X_∞^n the L_2- and L_4-norms are equivalent. Precisely,

$$\|y\|_2 \le \|y\|_4 \le (3^{1/4})\|y\|_2 \quad \text{for } y \in X_\infty^n . \tag{29}$$

Inequality (29) is well known in harmonic analysis, in much more general form, and it says that the y_j's form a Λ_4-set (cf., e.g., [R]). Its proof consists of a straightforward calculation evaluating the fourth power $\|y\|_4^4$ and depends on the fact that every integer $0 < l \le N-1$ can be represented as a difference $l = d_j - d_k$ in at most one way. As a standard consequence of (29), we get by Hölder's inequality

$$\|y\|_2 \le \|y\|_4^{2/3} \|y\|_1^{1/3} \le 3^{1/6} \|y\|_2^{2/3} \|y\|_1^{1/3} ,$$

so that

$$\|y\|_2 \le \sqrt{3} \|y\|_1 \quad \text{for } y \in X_\infty^n . \tag{30}$$

Now an argument from [Pis] shows that

$$\operatorname{ubc}(y_j) = \operatorname{ubc}(y_j \mid X_\infty^n) \le 3 \operatorname{gl}(X_\infty^n) . \tag{31}$$

Let us briefly describe how the argument works in our situation.

As in (c), consider the isometry $I : X_\infty^n \to X_\infty^n$ defined by

$$I\left(\sum_{j=1}^n a_j y_j\right) = \sum_{j=1}^n \exp\left(\frac{2\pi i}{N} d_j\right) a_j y_j \quad \text{for } \sum_{j=1}^n a_j y_j \in X_\infty^n .$$

In fact, I is the restriction to X_∞^n of the translation operator $\tilde{I} : \mathbf{C}^N \to \mathbf{C}^N$ defined by $\tilde{I}(\alpha_0, \dots, \alpha_{N-1}) = (\alpha_1, \dots, \alpha_{N-1}, \alpha_0)$. For $0 \le s \le N-1$ put

$$w^s = I^s w \quad \text{for } w \in X_\infty^n ; \qquad (w^*)^s = (I^*)^s w^* \quad \text{for } w^* \in (X_\infty^n)^* .$$

Clearly, $\|w^s\|_\infty = \|w\|_\infty$ and $\|(w^*)^s\|_\infty^* = \|w^*\|_\infty^*$ (here $\|\cdot\|_\infty^*$ denotes the norm on $(X_\infty^n)^*$).

Fix arbitrary $(\delta_j) \subset \mathbf{C}$ with $|\delta_j| = 1$ and $(a_j), (b_j) \subset \mathbf{C}$ for $j = 1, \dots, n$. Let $w := \sum_{j=1}^n a_j y_j \in X_\infty^n$ and define $w^* \in (X_\infty^n)^*$ by $\langle w^*, y_j\rangle = b_j$. Consider $A : (X_\infty^n)^* \to X_2^n$ and $B : X_\infty^n \to X_2^n$ given by

$$Ax^* = \sum_{j=1}^n \langle x^*, y_j\rangle \delta_j a_j y_j \quad \text{for } x^* \in (X_\infty^n)^*,$$

$$Bx = \sum_{j=1}^n b_j c_j y_j \quad \text{for } x = \sum_{j=1}^n c_j y_j \in X_\infty^n.$$

We will estimate $\pi_1(A)$ and $\pi_1(B)$. By the orthogonality of the y_j's and (30),

$$\begin{aligned}\|Ax^*\|_2 &= \left(\sum_{j=1}^n |\langle x^*, y_j\rangle a_j|^2\right)^{1/2} = \left\|\sum_{j=1}^n \langle x^*, y_j\rangle a_j y_j\right\|_2 \\ &\le \sqrt{3}\left\|\sum_{j=1}^n \langle x^*, y_j\rangle a_j y_j\right\|_1 = \sqrt{3}\frac{1}{N}\sum_{s=0}^{N-1} |\langle x^*, w^s\rangle|.\end{aligned}$$

Since $\|w^s\|_\infty = \|w\|_\infty$, this easily implies $\pi_1(A) \le \sqrt{3}\|w\|_\infty$. For B a similar calculation yields $\pi_1(B) \le \sqrt{3}\|w^*\|_\infty^*$. Namely

$$\begin{aligned}\|Bx\|_2 &= \left(\sum_{j=1}^n |b_j c_j|^2\right)^{1/2} = \left\|\sum_{j=1}^n b_j c_j y_j\right\|_2 \\ &\le \sqrt{3}\left\|\sum_{j=1}^n b_j c_j y_j\right\|_1 = \sqrt{3}\frac{1}{N}\sum_{s=0}^{N-1} |\langle (w^*)^s, x\rangle|.\end{aligned}$$

Since by trace-duality one has, with $\gamma_1(B) = \gamma_\infty(B^*)$,

$$\begin{aligned}\mathrm{gl}(X_\infty^n) &= \sup\{\gamma_\infty(B^*)/\pi_1(B) \mid 0 \ne B : X_\infty^n \to l_2^n\} \\ &= \sup\left\{\frac{|\mathrm{tr}(AB^*)|}{\pi_1(A)\pi_1(B)}\,\middle|\, 0 \ne A : (X_\infty^n)^* \to l_2^n,\ 0 \ne B : X_\infty^n \to l_2^n\right\},\end{aligned}$$

then

$$\begin{aligned}\left|\left\langle w^*, \sum_{j=1}^n \delta_j a_j y_j\right\rangle\right| &= \left|\sum_{j=1}^n \delta_j a_j b_j\right| = \mathrm{tr}(AB^*) \\ &\le \mathrm{gl}(X_\infty^n)\pi_1(A)\pi_1(B) \le 3\,\mathrm{gl}(X_\infty^n)\|w\|_\infty\|w^*\|_\infty^*.\end{aligned}$$

Passing to the supremum over $\|w^*\|_\infty^* \le 1$ and then over $|\delta_j| = 1$ and $\|w\|_\infty \le 1$, we get $\mathrm{ubc}(y_j \mid X_\infty^n) \le 3\,\mathrm{gl}(X_\infty^n)$, completing the proof of (31). Now the conclusion follows from (31) and (d). □

Since the space X_∞^n is translation invariant, statement (a) can be derived directly from (27), since P is the orthogonal projection. On the other hand, it seems that condition (b) cannot be proved without a general argument of Lemma 6.

Observe that the John's ellipsoid on X_∞^n is actually the unit ball from X_2^n. This follows immediately from the uniqueness of the distance inner product, established in (c), and the fact that the John's ellipsoid actually determines this distance.

Actually, as stated in Remark 4, the gl-constant of an n-dimensional space X_n satisfies $\mathrm{gl}(X_n) \le \sup_{Y_n} \lambda(Y_n) \le \sqrt{n} - c/\sqrt{n}$. This is easily seen by taking the 1-summing factorization of any operator $u : X_n \to l_2^n$.

$$\begin{tikzcd}
X_n \arrow[r, "u"] \arrow[d, "i"'] & l_2^n \arrow[r, "j"] & l_\infty \\
X_\infty^n \arrow[r, "I|X_\infty^n"] \arrow[d, phantom, "\cap"] & X_1^n \arrow[u, "\bar{u}"'] \arrow[d, phantom, "\cap"] & \\
L_\infty(\mu) \arrow[r, "I"'] & L_1(\mu) \arrow[uur, "v"'] &
\end{tikzcd}$$

Here $\pi_1(I) = 1$, and $\|v\| = \|\bar{u}\| = \pi_1(u)$. Projecting from $L_1(\mu)$ onto the subspace X_1^n with $\|P\| = \lambda(X_1^n) \le \sqrt{n} - c/\sqrt{n}$ yields an $L_1(\mu)$-factorization of $u : X_n \to l_2^n$. Thus

$$\mathrm{gl}(X_n) = \sup\{\gamma_1(u)/\pi_1(u) \mid u \ne 0\} \le \lambda(X_1^n) \le \sqrt{n} - c/\sqrt{n}.$$

Remark 6. It is not difficult to see that the type 2 constants of the spaces X_∞^n of Proposition 5 are of order $\sqrt{\ln n}$ and the (Gaussian) cotype 2 constants are of order $\sqrt{n/\ln n}$. An invariant closely related to the Gordon–Lewis constant was introduced in [PS] and called the Ké-constant. An argument somewhat similar to Proposition 5(f) shows that the Ké-constants of X_∞^n are between $\sqrt{n}/3$ and $\sqrt{n}$.

To our best knowledge, the spaces X_∞^n provide the first deterministic construction of spaces with the gl-constants of maximal order. Examples known before consist of random subspaces of l_∞^n, with N proportional to n (cf., e.g., [FJ], also [TJ, §34]).

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